Arithmetic and Geometric Progression

1. The geometric series converges if $\left|\frac{x}{1-x}\right| < 1 \Leftrightarrow |x| < |1-x| \Leftrightarrow x^2 < (1-x)^2 = 1-2x+x^2 \Leftrightarrow x < \frac{1}{2}$

The sum to infinity for such values of x = $\frac{1}{1-\left(-\frac{x}{1-x}\right)} = 1-x$.

2. The first series converges if $|x| < 1 \Leftrightarrow -1 < x < 1$ …..(1)

The second series converges if $\left|\frac{x}{1+x}\right| < 1 \Rightarrow -\frac{1}{2} < x$ …..(2)

From (1) and (2), the common range of x is $-\frac{1}{2} < x < 1$.

In this range, $s_1 = \frac{1}{1+x}$, $s_2 = \frac{1}{1-\frac{x}{1+x}} = 1+x$, $s_1 s_2 = 1$.

3. Let a be the first term and d the common difference.

$\therefore$ $a + (p-1)d = P$ (1) $a + (q-1)d = Q$ (2) $a + (r-1)d = R$ (3)

If $d = 0$, then $P = Q = R = a$ and the result is obvious.

If $d \neq 0$, then {(2) –(3)}/d, $q - r = (Q - R)/d$ (4)

{(3) –(1)}/d, $r - p = (R - P)/d$ (5)

{(1) –(2)}/d, $p - q = (P - Q)/d$ (6)

From (4), (5), (6), $P(q-r) + Q(r-p) + R(p-q) = \frac{P(Q-R)}{d} + \frac{Q(R-P)}{d} + \frac{R(P-Q)}{d} = 0$

If the sequence is geometrical, let a be the first term and b be the common ratio.

$\therefore$ $ab^{p-1} = P \Rightarrow \log a + (p-1)\log b = \log P$ (7)

$ab^{q-1} = P \Rightarrow \log a + (q-1)\log b = \log Q$ (8)

$ab^{r-1} = P \Rightarrow \log a + (r-1)\log b = \log R$ (9)

$\therefore$ log P, log Q, log R are the p^{th} , q^{th}, r^{th} terms respectively of an arithmetic sequence.

By the first part of this question, $(q-r)\log P + (r-p)\log Q + (p-q)\log R = 0$.

4. Let r be the common ratio. $1 + r + \ldots + r^{n-1} = 255$ (1)

$$1 + \frac{1}{r} + \ldots + \frac{1}{r^{n-1}} = \frac{255}{128} \quad (2)$$

From (2), $\frac{1}{r^{n-1}}\left(1 + r + \ldots + r^{n-1}\right) = \frac{255}{128}$ $\Rightarrow \frac{1}{r^{n-1}}(255) = \frac{255}{128}$ $\Rightarrow r^{n-1} = 128$ (3)

From (1), $\frac{r^{n-1}-1}{r-1} + r^{n-1} = 255$ $\Rightarrow \frac{128-1}{r-1} + 128 = 255$ $\Rightarrow r = 2$ (4)

(4)↓(3), $2^{n-1} = 128 = 2^7 = 2^{8-1}$. $\therefore$ $n = 8$.

5. "convergent series" – Let $u_1, u_2, \ldots, u_n, \ldots$ be an infinite sequence

and let $s_1 = u_1$, $s_2 = u_1 + u_2$, …, $s_n = u_1 + u_2 + \ldots + u_n$ be the partial sums. Then $u_1 + u_2 + \ldots + u_n + \ldots$

is a convergent series if $\lim_{n\to\infty} s_n = s$ exists. i.e. $\forall \varepsilon > 0 , \exists N(\varepsilon) > 0$ s.t. $n > N \Rightarrow |s_n - s| < \varepsilon$.

s is called the "sum to infinity of the convergent series".

For the geometric series : $s = a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$, first we prove that $\lim_{n\to\infty} r^n = 0$ when $|r| < 1$.

By the Bernoulli's inequality which states that if $1 + h > 0$ and $n \geq 2$, then $(1 + h)^n > 1 = nh$.

Now, since $|r| < 1$, we can write $|r| = \frac{1}{1+h}$, $h > 0$. Therefore $|r^n| = \frac{1}{(1+h)^n} < \frac{1}{1+nh}$.

Given $\varepsilon > 0$, $|r^n| < \varepsilon$ if $1 + nh > 1/\varepsilon$. Thus, $|r^n| < \varepsilon$ if $n > (1/\varepsilon - 1)/h$

Put $N = [(1/\varepsilon - 1)/h] + 1$, then $\forall \varepsilon > 0 , \exists N(\varepsilon) > 0$ s.t. $n > N \Rightarrow |r^n - 0| < \varepsilon$.

$\therefore \lim_{n\to\infty} r^n = 0$ when $|r| < 1$.

Obverve that : $s_n = a\frac{1-r^n}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$ $\therefore$ $\lim_{n\to\infty} s_n = \frac{a}{1-r}$, since $\frac{ar^n}{1-r} \to 0$

Note : When $r > 1$, $\lim_{n\to\infty} r^n = \infty$, the series diverges.

When $r = 1$, $s_n = na$, $\lim_{n\to\infty} s_n = \infty$, the series diverges.

When $r = -1$, $\lim_{n\to\infty} r^n$ oscillates between 1 and -1, the series oscillates finitely.

When $r < -1$, $\lim_{n\to\infty} r^n = \pm\infty$ the series oscillates infinitely.

(a) $2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \ldots$ converges since $|r| = 2/3 < 1$.

(b) $1 - 1 + 1 - 1 + 1 - \ldots$ oscillates infinitely since $r = -1$.

6. $S_{4m} = 44\, S_m \Rightarrow \frac{u_1 + u_{4m}}{2}(4m) = 44 \times \frac{u_1 + u_m}{2}(m) \Rightarrow u_1 + u_{4m} = 11(u_1 + u_m) \Rightarrow u_1 + 24 = 11(u_1 + 4)$

$\therefore$ $u_1 = -2$ $u_m = u_1 + (m-1)d = 4$ $\therefore$ $(m-1)d = 4 - u_1 = 6$ …..(1)

$u_{4m} = u_1 + (4m-1)d = 24$ $\therefore$ $(4m-1)d = 24 - u_1 = 26$ …..(2)

(2)/(1), $(4m-1)/(m-1) = 26/6$ $\therefore$ $m = 10$

7. $S_n = \frac{2u_1 + (n-1)d}{2}(n) = 2n$ $\Rightarrow 2u_1 + (n-1)d = 4$ …..(1)

$S_{2n} = \frac{2u_1 + (2n-1)d}{2}(2n) = n$ $\Rightarrow 2u_1 + (2n-1)d = 1$ …..(2)

(2) – (1), $nd = -3$ …..(3)

$$S_{4n} = \frac{2u_1 + (4n-1)d}{2}(n) = 2\left[\frac{2u_1 + (2n-1)d}{2}(2n) + \frac{2nd}{2}(2n)\right] = 2[S_{2n} + 2n(nd)] = 2[n + 2n(-3)]$$

$= \underline{\underline{-10n}}$, by (3).

8. Sum = $\sum_{0}^{332}(3r+1)+\sum_{0}^{332}(3r+2) = 6\sum_{0}^{332} r + 3\sum_{0}^{332} 1 = 6\dfrac{332\times 333}{2} + 3(333) = 3(333)(333) = \underline{\underline{332667}}$

9. The first integer in the n^{th} group $= 1 + 2 + \ldots + (n-1) + 1 = \dfrac{n(n-1)}{2} + 1 = \underline{\underline{\dfrac{n^2-n+2}{2}}}$

$\therefore$ Sum of the integers of the n^{th} group $= \dfrac{n}{2}[2u_1+(n-1)d] = \dfrac{n}{2}\left[2\times\dfrac{n^2-n+2}{2}+(n-1)1\right] = \underline{\underline{\dfrac{n(n^2+1)}{2}}}$

10. $S = (2m+1) + (2m+3) + \ldots + (4m-1) = \dfrac{(2m+1)+(4m+1)}{2}m = \underline{\underline{3m^2}}$

Since $3m^2$ contains a factor 3, therefore S is divisible by 3 .

If m is even , put $m = 2n$, where $n \in \mathbf{N}$**,** then $S = 3(2n)^2 = 12n^2$, which is divisible by 12 .

11. Let the three positive numbers by $a-d, a, a+d$.

Then, $(a-d)^2 + a^2 + (a+d)^2 = 165$ …..(1)

$(a-d) + a + (a+d) = 21$ …..(2)

From (2), $a = 7$ and substitute this value in (1), $d = \pm 3$, giving the three numbers: 4, 7, 10.

12. a^2, b^2, c^2 form an arithmetic sequence $\Rightarrow b^2 - a^2 = c^2 - b^2 \Rightarrow (b-a)(b+a) = (c-b)(c+b)$

$\Rightarrow \dfrac{b-a}{b+c} = \dfrac{c-b}{a+b} \Rightarrow \dfrac{b-a}{(c+a)(b+c)} = \dfrac{c-b}{(a+b)(a+c)} \Rightarrow \dfrac{1}{c+a} - \dfrac{1}{b+c} = \dfrac{1}{a+b} - \dfrac{1}{a+c}$

$\Rightarrow \dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ form an arithmetic sequence.

13. Since arithmetic sequence has constant common difference, the points $P = (p, a), Q = (q, b), R = (r, c)$ are points on a straight line. $\therefore m_{PQ} = m_{QR} \Rightarrow \dfrac{a-b}{p-q} = \dfrac{b-c}{q-r} \Rightarrow (q-r)a + (r-p)b + (p-q)c = 0$.

14. $(p, q), (q, p), (m, a_m)$ are collinear, giving Gradient $= \dfrac{a_m - p}{m-q} = \dfrac{p-q}{q-p} = -1 \Rightarrow a_m = p + q - m$

Since $(p, a_p), (q, a_q), (p+q, a_{p+q})$ are collinear, giving $\dfrac{a_{p+q}-a_p}{p+q-p} = \dfrac{a_p - a_q}{p-q} \Rightarrow \dfrac{a_{p+q}-a_p}{2q} = \dfrac{a_p-a_q}{2(p-q)}$

$\Rightarrow \dfrac{1}{q}\left[\left(\dfrac{a_1+a_{p+q}}{2}\right)-\left(\dfrac{a_1+a_p}{2}\right)\right] = \dfrac{1}{p-q}\left[\left(\dfrac{a_1+a_p}{2}\right)-\left(\dfrac{a_1+a_q}{2}\right)\right] \Rightarrow \dfrac{1}{q}\left[\dfrac{S_{p+q}}{p+q}-\dfrac{S_p}{p}\right] = \dfrac{1}{p-q}\left[\dfrac{S_p}{p}-\dfrac{S_q}{q}\right]$

$\Rightarrow S_{p+q} = \dfrac{p+q}{p-q}(S_p - S_q) \Rightarrow S_{p+q} = \dfrac{p+q}{p-q}(q-p) \Rightarrow S_{p+q} = -(p+q)$

15. From No. 14, $S_{p+q} = \dfrac{p+q}{p-q}(S_p - S_q) = 0$

16. $S_m = \dfrac{a_1+a_m}{2}m,\ S_n = \dfrac{a_1+a_n}{2}n$ $\therefore \dfrac{S_m}{S_n} = \dfrac{m^2}{n^2} \Rightarrow \dfrac{a_1+a_m}{a_1+a_n} = \dfrac{m}{n} \Rightarrow \dfrac{2a_1+(m-1)d}{2a_1+(n-1)d} = \dfrac{m}{n} \Rightarrow a_1 = \dfrac{d}{2}$

$\therefore \dfrac{a_m}{a_n} = \dfrac{a_1+(m-1)d}{a_1+(n-1)d} = \dfrac{d/2+(m-1)d}{d/2+(n-1)d} = \dfrac{2m-1}{2n-1}$

17. Let $a_2 = d$, then $a_k = a_1 + (k-1)d = (k-1)d$

$$S = \frac{a_3}{a_2}+\frac{a_4}{a_3}+\dots+\frac{a_n}{a_{n-1}} - a_2\left(\frac{1}{a_2}+\frac{1}{a_3}+\dots+\frac{1}{a_{n-2}}\right) = \frac{2}{1}+\frac{3}{2}+\dots+\frac{n-1}{n-2}-\left(1+\frac{1}{2}+\dots+\frac{1}{n-3}\right)$$

$$=\left(\frac{2}{1}-1\right)+\left(\frac{3}{2}-\frac{1}{2}\right)+\dots+\left(\frac{n-2}{n-3}-\frac{1}{n-3}\right)+\frac{n-1}{n-2} = 1+1+\dots+1+\left(1+\frac{1}{n-2}\right) = n-3+\left(1+\frac{1}{n-2}\right)$$

$$= n-2+\frac{1}{n-2}$$

18. $$S = \frac{\sqrt{a_2}-\sqrt{a_1}}{a_2-a_1}+\frac{\sqrt{a_3}-\sqrt{a_2}}{a_3-a_2}+\dots+\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n-a_{n-1}} = \frac{1}{d}\left[\left(\sqrt{a_2}-\sqrt{a_1}\right)+\left(\sqrt{a_3}-\sqrt{a_2}\right)+\dots+\left(\sqrt{a_n}-\sqrt{a_{n-1}}\right)\right]$$

$$=\frac{\sqrt{a_n}-\sqrt{a_1}}{d} = \frac{a_n-a_1}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)} = \frac{(n-1)d}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)} = \frac{n-1}{\sqrt{a_n}+\sqrt{a_1}}$$

19. $S = a_1^2 - a_2^2 + a_3^2 - a_4^2 + \dots + a_{2k-1}^2 - a_{2k}^2 = (a_1 - a_2)(a_1 + a_2) + (a_3 - a_4)(a_3 + a_4) + \dots + (a_{2k-1} - a_{2k})(a_{2k-1} + a_{2k})$

$$= -d\,(a_1 + a_2 + a_3 + a_4 + \dots + a_{2k-1} + a_{2k}) = -d\,S_{2k} = -d\frac{a_1+a_{2k}}{2}(2k) = \frac{k}{2k-1}(a_1+a_{2k})[-d(2k-1)]$$

$$=\frac{k}{2k-1}(a_1+a_{2k})[a_1-(a_1+d(2k-1))] = \frac{k}{2k-1}(a_1+a_{2k})(a_1-a_{2k}) = \frac{k}{2k-1}\left(a_1^2-a_{2k}^2\right)$$

20. **(i)** $S(n+3) - 3S(n+2) + 3S(n+1) - S(n) = S(n+3) - S(n+2) - 2[S(n+2) - S(n+1)] + [S(n+1) - S(n)]$

$= d - 2d + d = 0$, where d is the common difference.

(ii) $$3[\,S(2n) - S(n)\,] = 3\left[\frac{2a_1+(2n-1)d}{2}(2n)-\frac{2a_1+(n-1)d}{2}(n)\right] = \frac{3n}{2}[4a_1+(2n-1)(2)d-2a_1n-(n-1)d]$$

$$=\frac{2a_1+(3n-1)d}{2}(3n) = S(3n)$$

21. $$S_n(S_{3n}-S_{2n}) = a_1\frac{1-r^n}{1-r}\left[a_1\frac{1-r^{3n}}{1-r}-a_1\frac{1-r^{2n}}{1-r}\right] = a_1\frac{1-r^n}{1-r}\left[a_1\frac{r^{2n}-r^{3n}}{1-r}\right] = a_1\frac{1-r^n}{1-r}r^n\left[a_1\frac{r^n-r^{2n}}{1-r}\right]$$

$$=\left[a_1\frac{r^n-r^{2n}}{1-r}\right]\left[a_1\frac{r^n-r^{2n}}{1-r}\right] = \left[a_1\frac{r^n-r^{2n}}{1-r}\right]^2 = \left[a_1\frac{1-r^{2n}}{1-r}-a_1\frac{1-r^n}{1-r}\right]^2 = (S_{2n}-S_n)^2 .$$

22. By the Lagrange's Identity (which can be proved by expansion) :

$(x_1^2 + x_2^2 + \dots + x_{n-1}^2)(y_1^2 + y_2^2 + \dots + y_{n-1}^2) - (x_1y_1 + x_2y_2 + \dots + x_{n-1}y_{n-1})^2$

$= (x_1y_2 - x_2y_1)^2 + (x_1y_3 - x_3y_1)^2 + \dots + (x_{n-2}y_{n-1} - x_{n-1}y_{n-2})^2$

Putting $x_1 = a_1$, $x_2 = a_2$, …, $x_{n-1} = a_{n-1}$; $y_1 = a_2$, $y_2 = a_3$, …, $y_{n-1} = a_n$

$(a_1^2 + a_2^2 + \dots + a_{n-1}^2)(a_2^2 + a_3^2 + \dots + a_n^2) = (a_1 a_2 + a_2 a_3 + \dots + a_{n-1} a_n)^2$

$\Leftrightarrow$ $(a_1^2 + a_2^2 + \dots + a_{n-1}^2)(a_2^2 + a_3^2 + \dots + a_n^2) - (a_1 a_2 + a_2 a_3 + \dots + a_{n-1} a_n)^2 = 0$

$\Leftrightarrow$ $(a_1a_3 - a_2^2)^2 + (a_1a_4 - a_3a_2)^2 + \dots + (a_{n-2}a_n - a_n^2)^2 = 0$

$\Leftrightarrow$ $a_1a_3 - a_2^2 = 0 \;\wedge\; a_1a_4 - a_3a_2 = 0 \;\wedge \dots \wedge\; a_{n-2}a_n - a_n^2 = 0$

$\Leftrightarrow$ $\frac{a_1}{a_2} = \frac{a_2}{a_3} = \dots = \frac{a_{n-1}}{a_n}$

$\Leftrightarrow$ $a_1, a_2, \dots, a_n$ form a geometric sequence.

23. Since $S_k = a_1 \dfrac{r^k - 1}{r-1} = \dfrac{a_{k+1} - a_1}{r-1}$

(i) $S_1 + S_2 + \dots + S_n = \dfrac{a_2 - a_1}{r-1} + \dfrac{a_3 - a_1}{r-1} + \dots + \dfrac{a_{n+1} - a_1}{r-1} = \dfrac{(a_1 + a_2 + \dots + a_{n+1}) - (n-1)a_1}{r-1}$

$$= \frac{S_{n+1}}{r-1} - \frac{(n-1)a_1}{r-1} = \frac{a_1 r^{n+1} - a_1}{(r-1)^2} - \frac{(n-1)a_1}{r-1} = a_1 \frac{r^{n+1} + n + r - nr}{(r-1)^2}$$

(ii) $\dfrac{1}{a_1^2 - a_2^2} + \dfrac{1}{a_2^2 - a_3^2} + \dots + \dfrac{1}{a_{n-1}^2 - a_n^2} = \dfrac{1}{1-r^2}\left[\dfrac{1}{a_1^2} + \dfrac{1}{a_2^2} + \dots + \dfrac{1}{a_{n-1}^2}\right] = \dfrac{1}{1-r^2}\dfrac{1}{a_1^2}\dfrac{\left(\dfrac{1}{r^2}\right)^{n-1} - 1}{\dfrac{1}{r^2} - 1}$

$$= \frac{1}{a_1^2}\frac{1}{(1-r^2)^2}\frac{1}{r^{2(n-2)}} = \frac{1}{a_1^2 r^{2(n-2)}(1-r^2)^2}$$

(iii) $\dfrac{1}{a_1^k + a_2^k} + \dfrac{1}{a_2^k + a_3^k} + \dots + \dfrac{1}{a_{n-1}^k + a_n^k} = \dfrac{1}{1+r^k}\left[\dfrac{1}{a_1^k} + \dfrac{1}{a_2^k} + \dots + \dfrac{1}{a_{n-1}^k}\right] = \dfrac{1}{1+r^k}\dfrac{1}{a_1^2}\dfrac{\left(\dfrac{1}{r^k}\right)^{n-1} - 1}{\dfrac{1}{r^k} - 1}$

$$= \frac{1}{1+r^k}\frac{1}{a_1^2}\frac{1}{r^{k(n-2)}}\frac{1-r^{k(n-1)}}{1-r^k} = \frac{1-r^{k(n-1)}}{a_1^2 r^{k(n-2)}(1-r^{2k})}$$

24. Let the arithmetic sequence be $a_1, a_2, \dots, a_n$. Let a_k^* be the k^{th} term from the end of the sequence.

$\therefore \quad a_k^* = a_n - (k-1)d, \qquad a_k = a_1 + (k-1)d$

$\therefore \quad a_k a_k^* = a_1 a_n - (k-1)^2 d^2 + (k-1)d(a_n - a_1) = a_1 a_n - (k-1)^2 d^2 + (k-1)(n-1)d^2$

$\qquad = a_1 a_n + d^2 \{(k-1)(n-1) - (k-1)^2\} = a_1 a_n + d^2 (k-1)(n-k) = a_1 a_n + d^2 P_k$

It only remain to show that P_k increases with an increase in k from 1 to $n/2$ or $(n+1)/2$.

Since $P_k = (k-1)(n-k)$, $P_{k+1} = k(n-k-1)$, therefore $P_{k+1} - P_k = n - 2k$

$\therefore \quad P_{k+1} > P_k$ if $n - 2k > 0$, i.e. if $k < n/2$.

25. Let the arithmetic sequence be $a_1, a_2, \dots, a_n$. $a_i > 0$.

Let the geometric sequence be $u_1, u_2, \dots, u_n$. $u_i > 0$.

Then $a_1 = u_1$, $a_n = u_n$. First we prove for $k = 0, 1, 2, \dots, (n-1)$, $u_{k+1} + u_{n-k} < u_1 + u_n$.

Proof : $u_{k+1} + u_{n-k} - (u_1 + u_n) = u_1 r^k + u_1 r^{n-k-1} - u_1 - u_1 r^{n-1} = u_1 (r^k - 1)(1 - r^{n-k-1}) < 0$

since $u_1 > 0$ and if $r > 1$, $r^k - 1 > 0$ and $1 - r^{n-k-1} < 0$

if $r < 1$, $r^k - 1 < 0$ and $1 - r^{n-k-1} > 0$.

Now, $u_{k+1} + u_{n-k} < u_1 + u_n \Rightarrow \displaystyle\sum_{k=0}^{n} (u_{k+1} + u_{n-k}) < \sum_{k=0}^{n} (u_1 + u_k)$

$\therefore \quad 2(u_1 + u_2 + \dots + u_n) < n(u_1 + u_n) = n(a_1 + a_n) = (a_1 + a_n) + (a_1 + a_n) + \dots + (a_1 + a_n)$

$\qquad = (a_1 + a_n) + (a_2 + a_{n-1}) + \dots + (a_n + a_1) = 2(a_1 + a_2 + \dots + a_n)$

since $a_{k+1} + a_{n-k} = (a_1 + kd) + [a_1 + (n-k-1)d] = a_1 + [a_1 + (n-1)d] = a_1 + a_n$.

$\therefore \quad u_1 + u_2 + \dots + u_n < a_1 + a_2 + \dots + a_n$

26. Let the first common term of the sequences be a and the second term be b.

Then the n^{th} term of the arithmetic sequence is $a + (b - a)(n - 1)$ and the corresponding term of the geometric sequence is $a(b/a)^{n-1}$.

$$a+(b-1)(n-1)-a\left(\frac{b}{a}\right)^{n-1} = a\left\{\left(\frac{b}{a}-1\right)(n-1)-\left[\left(\frac{b}{a}\right)^{n-1}-1\right]\right\}$$

$$= a\left(\frac{b}{a}-1\right)\left\{(n-1)-\left[\left(\frac{b}{a}\right)^{n-2}+\left(\frac{b}{a}\right)^{n-3}+\ldots+\left(\frac{b}{a}\right)+1\right]\right\} = a\left(\frac{b}{a}-1\right)\left\{\left[1-\left(\frac{b}{a}\right)^{n-2}\right]+\left[1-\left(\frac{b}{a}\right)^{n-3}\right]+\ldots+\left[1-\left(\frac{b}{a}\right)\right]\right\}$$

≤ 0 by considering separately the three cases : $\frac{b}{a} > 1$, $\frac{b}{a} = 1$, $\frac{b}{a} < 1$.

$\therefore$ $a+(b-1)(n-1) \leq a\left(\frac{b}{a}\right)^{n-1}$ and the proposition is proved.

27. $a_1a_3 \ldots a_{2n-1}$ $= a_1 (a_1 r^2) (a_1 r^4) \ldots (a_1 r^{2n-2})$

$= a_1^n (1)(r^2) (r^4) \ldots (r^{2n-2})$

$= a_1^n r^{[2+4+\ldots+(2n-2)]}$

$= a_1^n r^{n(n-1)}$.

$a_2a_4\ldots a_{2n}$ $= (a_1 r) (a_1 r^3) \ldots (a_1 r^{2n-1})$

$= a_1^n (r) (r^3) \ldots (r^{2n-1})$

$= a_1^n r^{[1+3+\ldots+(2n-1)]}$

$= a_1^n r^{(n^2)}$